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A perturbative approach to a class of Fokker-Planck equations

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In this paper we present a direct perturbative method to solving certain Fokker-Planck equations, which have constant diffusion coefficients and some small parameters in the drift coefficients. The method makes use of the connection between the Fokker-Planck and Schrödinger equations. Two examples are used to illustrate the method. In the first example the drift coefficient depends only on time but not on space. In the second example we consider the Uhlenbeck-Ornstein process with a small drift coefficient. These examples show that the such perturbative approach can be a useful tool to obtain approximate solutions of Fokker-Planck equations with constant diffusion coefficients.

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I. INTRODUCTION

The Fokker-Planck (FP) equation is one of the basic tools used to deal with fluctuations in various kinds of systems [1]. It has found applications in such diverse areas as physics, astrophysics, chemistry, biology, finance, etc. Owing to its wide applicability, various methods of finding exact and approximate solutions of the FP equations have been developed. Most of the methods, however, are concerned with FP equations with time-independent diffusion and drift coefficients [1]. Generally, it is not easy to find solutions of FP equations with time-dependent diffusion and drift coefficients.

One of the methods of solving FP equation with time-independent diffusion and drift coefficients is to transform the FP equation into a Schrödinger-like equation, and then solve the eigenvalue problem of the latter. The transformation to the Schrödinger equation of a FP equation eliminates the first order spatial derivative in the FP operator and creates a Hermitian spatial differential operator. This method is useful when the associated Schrödinger equation is exactly solvable; for example with infinite square well, harmonic oscillator potentials, etc. Several FP equations have been exactly solved in this way [1]. The method can also be useful for approximate results if a finite part of the spectrum of the associated Schrödinger equation can be exactly known [2]. When this happens the Schrödinger equation, and hence the associated FP equation, are called quasi-exactly solvable [3].

In this paper we would like to show that the connection between the FP and Schrödinger equations can also be useful even when the drift coefficients are time-dependent. Based on this connection, we present a direct perturbative method which can be used to solve a certain class of FP equations.

II. PERTURBATIVE APPROACH TO FP EQUATIONS

In one dimension, the FP equation of the probability density $P(x,t)$ is [1]

$$\frac{\partial}{\partial t}W(x,t) = \left(-\frac{\partial}{\partial x}D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2}D^{(2)}(x,t)\right)W(x,t). \tag{1}$$

The functions $D^{(1)}(x,t)$ and $D^{(2)}(x,t)$ in the FP equation are, respectively, the drift and the diffusion coefficient. The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. The drift coefficient is usually expressed in terms of a drift potential $U(x,t)$ according to $D^{(1)}(x,t) = -\partial U(x,t)/\partial x$. In this paper we shall be concerned with an important class of FP equations, namely, those with constant diffusion coefficients $D^{(2)}(x,t) = D > 0$.

The FP equation is closely related to the Schrödinger equation. A FP equation with time-independent drift $D^{(1)}(x,t) = D^{(1)}(x)$ can be transformed via a similarity transformation to a Schrödinger-like equation with time-independent potential [1, 2, 5]. Hence the FP equation can be exactly solved if the associated Schrödinger equation can be exactly solved. It is also due to this connection that time-independent WKB method can be applied to obtain approximate solutions of the FP equation [4].

When the drift coefficient of the FP equation is time-dependent, the FP equation can still be transformed into a time-dependent Schrödinger-like equation, only now the potential in its associated Schrödinger equation is time-dependent. We show their connection below.

We first define $\psi(x,t) \equiv e^{U(x,t)/2D}W(x,t)$. Substituting this into the FP equation, we find that ψ satisfies the Schrödinger-like equation:

$$\frac{\partial\psi}{\partial t} = D\frac{\partial^2\psi}{\partial x^2} + \left(\frac{U''}{2} - \frac{U'^2}{4D} + \frac{\dot{U}}{2D}\right)\psi, \tag{2}$$

where the prime and dot denote the derivatives with respect to x and t, respectively. Eq. (2) is the time-dependent Schrödinger-like equation associated with the FP equation. Owing to the $\dot{U}$ term, the potential in the Schrödinger equation is time-dependent.

If (2) can be exactly solved, then the original FP equation is also exactly solved. However, solving the Schrödinger equations with time-dependent potentials is generally difficult. In most cases one has to resort to approximate or numerical methods.

In this paper, we would like to show that a direct perturbative approach can be useful to solve the FP equation containing a small parameter in the drift potential.

We shall consider drift potential of the form $U(x,t) = \sum_{n=0}^{\infty}\lambda^n U_n(x,t)$, where $|\lambda| << 1$ is a small parameter. We introduce the ansatz $\psi = \exp(S(x,t,\lambda)/D)$, where S is expanded in powers of λ:

$$S(x,t,\lambda) = \sum_{n=0}^{\infty}\lambda^n S_n(x,t). \tag{3}$$

From (2) we find that $S(x,t,f)$ satisfies

$$\dot{S} = DS'' + S'^2 + \overline{U}, \tag{4}$$

$$\overline{U} \equiv \frac{D}{2}U'' - \frac{1}{4}U'^2 + \frac{1}{2}\dot{U}. \tag{5}$$

Substituting (3) into (4) and collecting terms of the same order in λ, we arrive at a set of differential equations determining the functions S_n:

$$\dot{S}_0 = DS_0'' + S_0'^2 + \frac{D}{2}U_0'' - \frac{1}{4}U_0'^2 + \frac{1}{2}\dot{U}_0, \tag{6}$$

$$\dot{S}_1 = DS_1'' + 2S_0'S_1' + \text{ terms in } \overline{U} \text{ of the order } \lambda, \tag{7}$$

$$\dot{S}_2 = DS_2'' + 2S_0'S_2' + S_1'^2 + \text{ terms in } \overline{U} \text{ of the order } \lambda^2, \tag{8}$$

$$\vdots$$

$$\dot{S}_n = DS_n'' + \sum_{k=0}^{n} S_k'S_{n-k}' + \text{ terms in } \overline{U} \text{ of the order } \lambda^n, \quad n \geq 0. \tag{9}$$

We note that the probability density $W_0 = \exp(S_0/D)$ with S_0 satisfying (6) is the solution of the FP equation with constant D and drift potential $U_0(x,t)$.

By solving S_0, S_1, $S_2, \ldots$ from the above equations, we can obtain an approximate solution of ψ, and hence of $W(x,t)$. We shall illustrate this by two simple examples below.

III. FP EQUATIONS WITH DRIFT POTENTIAL $U(x,t) = \lambda x V(t)$

Let us consider a class of FP equation with a weak drift potential of the form $U(x,t) = \lambda x V(t)$, where $V(t)$ is some finite function of time and $|\lambda| << 1$. In this case, $U_0 = U_n = 0$ $(n \geq 2)$. This FP equation can thus be viewed as a diffusion equation (or Wiener process) perturbed by a weak time-dependent drift potential.

Let us assume the initial profile of W to be the delta-function, i.e. $W(x,t) \to \delta(x)$ at $t = 0$ as $\lambda \to 0$. Then the solution of (6) is

$$S_0(x,t) = -\frac{D}{2}\ln(4\pi Dt) - \frac{x^2}{4t}, \tag{10}$$

giving the probability density

$$W_0 = e^{\frac{S_0}{D}} = \frac{1}{\sqrt{4\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{11}$$

We have $\int_{-\infty}^{\infty} W_0 dx = 1$. This is the well known solution of the diffusion equation having the delta-function as its initial profile.

$S_1(x,t)$ is determined from S_0 by (7)

$$\dot{S}_1 = DS_1'' - \frac{x}{t}S_1' + \frac{x}{2}\dot{V}. \tag{12}$$

It is easily checked that (12) is solved by

$$S_1(x,t) = \frac{x}{2}V(t) - \frac{x}{2t}\overline{V}(t) + c_0 \tag{13}$$
$$\overline{V}(t) \equiv \int^t V(t)dt + c_1. \tag{14}$$

Here c_0 and c_1 are arbitrary real constants. c_0 can be absorbed into the normalization constant of $W(x,t)$. We demand that c_1 be so chosen as to render S_1 finite in the limit $t \to 0$. For instance, if $V(t) = \cos(\omega t)$, we shall choose $c_1 = 0$, whilst for $V(t) = \sin(\omega t)$ we have $c_1 = 1/\omega$.

For $S_2(x,t)$, it is determined by (8)

$$\dot{S}_2 = DS_2'' - \frac{x}{t}S_2' + \frac{1}{4}\left(V(t) - \frac{1}{t}\overline{V}(t)\right)^2 - \frac{V(t)^2}{4}. \tag{15}$$

Solution of (15) is

$$S_2(x,t) = -\frac{1}{4t}\overline{V}(t)^2 + c_3\frac{x}{t} + c_2. \tag{16}$$

Again, c_2 can be absorbed into the normalization constant. We demand that S_2 be finite for all x as $t \to 0$, hence we set $c_3 = 0$. With these forms of S_0, S_1 and S_2, one finds from (9) that S_3 $(n \geq 3)$ satisfies the equation:

$$\dot{S}_3 = DS_3'' - \frac{x}{t}S_3', \tag{17}$$

which is solved by $S_3 = c_5 x/t + c_4$ with arbitrary constants c_4 and c_5. Again finiteness of S_3 as $t \to 0$ rules out the x/t term, and the constant term can be absorbed into the normalization constant, and hence we set $c_4 = c_5 = 0$. Continuing this process, we find that all subsequent S_n's also satisfy (17), and hence all S_n must be taken as constant, which can all be set to zero.

Hence, in this example we can solve all the S_n from the set of equations (7), (8) to (9). The probability density $W(x,t)$ is given by the following expression:

$$\begin{aligned} W(x,t) &= e^{-\frac{\lambda x}{2D}V(t)} e^{(S_0 + \lambda S_1 + \lambda^2 S_2)/D} \\ &= \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{1}{4Dt}\left(x + \lambda\overline{V}(t)\right)^2\right). \end{aligned} \tag{18}$$

W in (18) is normalized. One notes that (18) is very similar to the solution (11) of the diffusion equation, only with the x in W_0 being replaced by $\bar{x} \equiv x + \lambda\overline{V}(t)$. It turns out that W in (18) is indeed the exact solution of the FP equation with $U(x,t) = \lambda x V(t)$ for any value of λ. In fact, in terms of the new variable $\bar{x}$ and t, the FP equation reduces to the diffusion equation:

$$\frac{\partial W(\bar{x},t)}{\partial t} = D\frac{\partial^2 W(\bar{x},t)}{\partial \bar{x}^2}. \tag{19}$$

Eq. (18) is the exact solution of this diffusion equation with initial profile $\delta(x + \lambda\overline{V}(0))$. The peak of $W(x,t)$ shifts according to the function $\lambda\overline{V}(t))$. For instance, if $V(x) \sim \sin(\omega t)$ or $\cos(\omega t)$, the peak oscillates about $x = 0$ as $W(x,t)$ spreads outward.

IV. UHLENBECK-ORNSTEIN PROCESS WITH A SMALL DRIFT POTENTIAL

Next we consider the Uhlenbeck-Ornstein process [1, 6], which is described by the FP equation with constant diffusion coefficient and a time-independent drift potential $U(x) = \lambda x^2/2$. This process is exactly solvable, with the probability density given by

$$W(x,t) = \sqrt{\frac{\lambda}{2\pi D(1 - e^{-2\lambda t})}} \exp\left(-\frac{\lambda x^2}{2D(1 - e^{-2\lambda t})}\right). \tag{20}$$

We would like to apply the above perturbative approach to this process when λ is small. Hence we are treating the Uhlenbeck-Ornstein process as a perturbed Wiener process.

The associated Schrödinger equation (2) and $\overline{U}$ in (5) do not contain $\dot{U}$ in this case. S_0 is still to be solved from (6). If we assume an initial profile $W(x,0) \to \delta(x)$ as $\lambda \to 0$, then S_0 is again given by (10).

With this S_0, (7) and (8) read

$$\dot{S}_1 = DS_1'' - \frac{x}{t}S_1' + \frac{D}{2}, \tag{21}$$

$$\dot{S}_2 = DS_2'' - \frac{x}{t}S_2' + S_1'^2 - \frac{x^2}{4}. \tag{22}$$

Eq. (21) is solved by

$$S_1 = \frac{1}{2}Dt + c_1\frac{x}{t} + c_0, \quad c_0,\ c_1\text{: arbitrary constants.} \tag{23}$$

As before, finiteness of S_1 requires $c_1 = 0$, and c_0 is absorbed into normalization constant. Plugging $S_1 = Dt/2$ into (22), we obtain the solution for S_2:

$$S_2 = -\frac{1}{12}Dt^2 - \frac{1}{12}x^2t + c_3\frac{x}{t} + c_2, \quad c_2,\ c_3\text{: arbitrary constants.} \tag{24}$$

By the same reasons given before, we set $c_3 = 0$ and leave out c_2. Putting all the solutions together, we have, up to the λ^2 terms, the approximate expression for W

$$W(x,t) \approx \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\left(1 + \lambda t + \frac{1}{3}\lambda^2t^2\right) + \frac{1}{2}\lambda t - \frac{1}{12}\lambda^2t^2\right). \tag{25}$$

Now using the series expansion of $\ln(1+x) \approx x - x^2/2 + \ldots$ for small x, we can rewrite the last two terms in the exponent of (25) as

$$\frac{1}{2}\lambda t - \frac{1}{12}\lambda^2t^2 \approx \frac{1}{2}\ln\left(1 + \lambda t + \frac{1}{3}\lambda^2t^2\right). \tag{26}$$

Then (25) becomes

$$W(x,t) \approx \sqrt{\frac{1 + \lambda t + \frac{1}{3}\lambda^2t^2}{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\left(1 + \lambda t + \frac{1}{3}\lambda^2t^2\right)\right). \tag{27}$$

W given by (27) is normalized. It is easily checked that (27) is the same as the approximate expression obtained from (20) when the factors in the square-root and the exponent are expanded up to λ^2 terms. Hence, the result obtained by the perturbative approach is consistent with the exact result.

V. SUMMARY

In summary, we have applied a direct perturbative theory to solve certain FP equations, which have constant diffusion coefficients. Two examples are used to illustrate the method. In the first example the drift coefficient depends only on time but not on space. In the second example we treat the Uhlenbeck-Ornstein process with a small drift coefficient. These examples demonstrate that such perturbative approach is feasible for obtaining approximate solutions of FP equations.

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